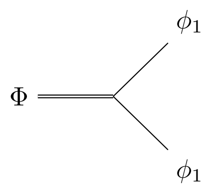
Quantum Field Theory

Solutions of Training Exercises

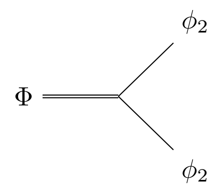
Exercise 1

Part 1

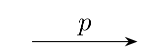
- Each term of the Lagrangian has energy dimension 4. Scalar fields have one dimension of energy (which can be checked by looking at their kinetic term, where the derivatives have dimension 1). Then, looking at the interaction terms containing 3 scalar fields, we deduce the couplings have energy dimension 1.
- We have the follwing Feynman rules:



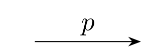
$= -i\lambda_1$



$= -i\lambda_2$ (1)



$= \dfrac{i}{p^2 + i\varepsilon}$

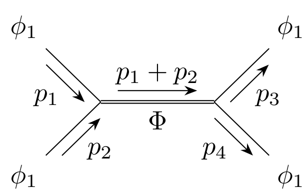


$= \dfrac{i}{p^2 - M^2 + i\varepsilon}$ (2)

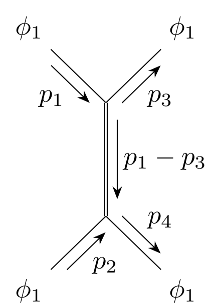
Notice that there are no factors $\frac{1}{2}$ in the vertices, because we have already taken into account the two ways to connect the ϕ_i fields.

Let's compute in detail the $\phi_1\phi_1 \to \phi_1\phi_1$ amplitude. There are three diagrams, which correspond to the usual s,t,u-channel:

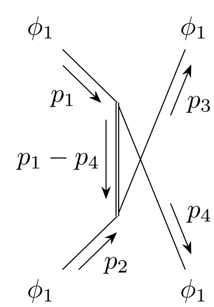
$i\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1) =$



$+$



$+$



. (3)

We simply get

$$i\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1) = (-i\lambda_1)^2 \frac{i}{(p_1+p_2)^2 - M^2} + (-i\lambda_1)^2 \frac{i}{(p_1-p_3)^2 - M^2} + (-i\lambda_1)^2 \frac{i}{(p_1-p_4)^2 - M^2}\,, \tag{4}$$

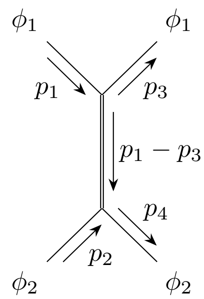
$$\mathcal{M}(\phi_1\phi_1 \to \phi_1\phi_1) = -\lambda_1^2 \left(\frac{1}{s - M^2} + \frac{1}{t - M^2} + \frac{1}{u - M^2} \right)\,. \tag{5}$$

We get something similar for the $\phi_2\phi_2 \to \phi_2\phi_2$ process, only the coupling at the vertices changes

$$\mathcal{M}(\phi_2\phi_2 \to \phi_2\phi_2) = -\lambda_2^2 \left(\frac{1}{s - M^2} + \frac{1}{t - M^2} + \frac{1}{u - M^2} \right)\,. \tag{6}$$

The $\phi_1\phi_2 \to \phi_1\phi_2$ is a bit different. Because there is no $\phi_1\phi_2\Phi$ interaction, here we have only one diagram:

$$i\mathcal{M}(\phi_1\phi_2 \to \phi_1\phi_2) = \quad (7)$$



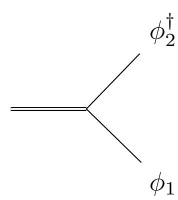
So
$$\mathcal{M}(\phi_1\phi_2 \to \phi_1\phi_2) = \frac{-\lambda_1\lambda_2}{t - M^2}\,. \quad (8)$$

- Clearly the two first amplitudes are equal if and only if $\lambda_1 = \lambda_2$. In this case, the interaction term in the lagrangian is
$$\mathcal{L}_{int} = -\lambda_1\Phi(\phi_1^2 + \phi_2^2) = -\lambda_1\Phi\phi^2\,, \quad (9)$$
where we have defined a 2-components field $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$. We see that we have more symmetry to this lagrangian, it is now invariant under $O(2)$ transformations of ϕ (leaving Φ invariant). This symmetry is the reason why these processes do not distinguish between ϕ_1 and ϕ_2. Then, observing the third process, which has a quite different amplitude, could be a way to prove there exist indeed two fields and not just one.

Part 2

- All diagrams are simple interactions. For example

$$i\mathcal{M}(\Phi_1 \to \phi_2^\dagger\phi_1) \;=\; \Phi_1$$



$$= -i\lambda_1 \quad (10)$$

Thus
$$d\Gamma(\Phi_1 \to \phi_2^\dagger\phi_1) = \frac{1}{2M}\left(\prod_{i=1,2}\frac{d^3p}{(2\pi)^3}\frac{1}{2E_i}\right)|\mathcal{M}(\Phi_1 \to \phi_2^\dagger\phi_1)|^2(2\pi)^4\delta(4)(P_A - p_1 - p_2)\,. \quad (11)$$

Then, using the formula for 2-body phase space proven in class
$$\Gamma(\Phi_1 \to \phi_2^\dagger\phi_1) = \int d\Gamma(\Phi_1 \to \phi_2^\dagger\phi_1) = \frac{1}{2M}\lambda_1^2\int\frac{d\Omega_{CM}}{4\pi}\frac{1}{8\pi}\frac{2|\vec{p}|}{E_{CM}} = \frac{\lambda_1^2}{16\pi M}\,. \quad (12)$$

The total energy is simply $E_{CM} = M$, and the magnitude of the 3-momentum of final particles is $|\vec{p}| = M/2$ since by conservation of 4-momentum $(|\vec{p}|, \vec{p}) + (|\vec{p}|, -\vec{p}) = (M, \vec{0})$.

Proceding very similarly, we find
$$\Gamma(\Phi_1 \to \phi_2\phi_1^\dagger) = \frac{\lambda_1^2}{16\pi M} \quad (13)$$

so the total width is
$$\Gamma(\Phi_1) = \frac{\lambda_1^2}{8\pi M}\,. \quad (14)$$

Φ_2 has the same decay channels, and we find the total rate
$$\Gamma(\Phi_2) = \frac{\lambda_2^2}{8\pi M}\,. \quad (15)$$

Φ_3 decays to $\phi_1\phi_1^\dagger$ or $\phi_2\phi_2^\dagger$. There is no difference in the computation and we have

$$\Gamma(\Phi_3) = \frac{\lambda_3^2}{8\pi M}\,. \tag{16}$$

- Clearly, if $\lambda_1 = \lambda_2 = \lambda_3$, the decay widths are the same.
- In this case, we can rewrite the interaction as

$$\mathcal{L}_{int} = -\lambda_1 \left(\Phi_1(\phi_2^\dagger\phi_1 + \phi_1^\dagger\phi_2) + \Phi_2(i\phi_2^\dagger\phi_1 - i\phi_1^\dagger\phi_2) + \Phi_3(\phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2) \right) = -\lambda_1 \sum_{A=1}^{3} \Phi_A \phi^\dagger \sigma^A \phi\,, \tag{17}$$

where we have introduced the 2-component field $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ and the Pauli matrices. Written in this way, the interaction term is now manifestly $SU(2)$-invariant if we define ϕ as transforming as a doublet, and Φ as a triplet. Indeed, if we consider a $SU(2)$ transformation defined by $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$, it acts as

$$\begin{aligned} \phi &\to \exp(i\theta_i\sigma_i/2)\phi = U_R\phi\,, \\ \Phi_A &\to \exp(i\theta_i J_i)_{AB}\Phi = R_{AB}\Phi_B\,, \end{aligned} \tag{18}$$

and using

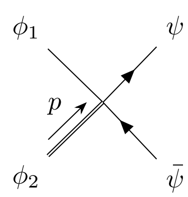
$$U_R^\dagger \sigma^A U_R = R_{AB}\sigma^B \tag{19}$$

We find that the lagrangian is invariant. As in the first part, the equality of decay rates is a consequence of an enhanced global symmetry.

- Yes, since the symmetry is exact, it will still guarantee that the Φ_A have the same decay rates if one computes them at any higher order in perturbation theory.

Exercise 2

- As before, scalars have dimension 1, and fermions have dimension $\frac{3}{2}$, as can be seen from their kinetic term. In the interaction term, the two scalars, two fermions and one derivative have total dimension 6, so the coupling has energy dimension -2.
- The Feynman rule for the vertex is the following:

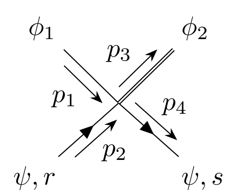


$$= -ig(-ip)_\mu\gamma^\mu = -g\,\not{p}\,, \tag{20}$$

where the i is added as usual to the $-g$ factor in the lagrangian, the γ^μ comes from the lagrangian, and the $(-ip)_\mu$ comes from the derivative in the lagrangian. Since the derivative is on the ϕ_2 field, only the momentum coming in through the ϕ_2 leg appears in the rule. It will change sign if the momentum on the ϕ_2 line is outgoing instead of incoming.

Let us compute scattering amplitudes

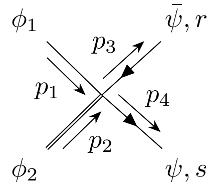
$$i\mathcal{M}(\phi_1(p_1)\psi(p_2,r) \to \phi_2(p_3)\psi(p_4,s)) =$$



$$= \bar{u}^s(p_4)g\,\not{p}_3 u^r(p_2)\,, \tag{21}$$

where, as usual we asociate $\bar{u}^s$ to an outgoing fermion of polarization s, u^r to an incoming fermion, and we reversed the sign of the vertex since p_3 is going out, as mentioned.

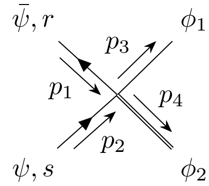
$$i\mathcal{M}(\phi_1(p_1)\phi_2(p_2) \to \bar{\psi}(p_3,r)\psi(p_4,s)) \ = $$



$$= \ -\bar{u}^s(p_4) g \not{p}_2 v^r(p_3)\,, \tag{22}$$

where this time we have used v^r for the outgoing antifermion.

$$i\mathcal{M}(\bar{\psi}(p_1,r)\psi(p_2,s) \to \phi_1(p_3)\phi_2(p_4)) \ = $$



$$= \ \bar{v}^r(p_1) g \not{p}_4 u^s(p_2)\,. \tag{23}$$

- The next step towards the unpolarized cross-section is to square the amplitude, average over initial polarizations and sum over final polarizations.

$$\begin{aligned}
\frac{1}{2}\sum_{r,s}|\mathcal{M}(\phi_1\psi \to \phi_2\psi)|^2 &= \frac{1}{2}\sum_{r,s}\bar{u}^s(p_4)g\not{p}_3 u^r(p_2)\big(\bar{u}^s(p_4)g\not{p}_3 u^r(p_2)\big)^* \\
&= \frac{g^2}{2}\sum_{r,s}\bar{u}^s(p_4)g\not{p}_3 u^r(p_2)\bar{u}^r(p_2)\not{p}_3 u^s(p_4) = \frac{g^2}{2}\,\mathrm{tr}\left[(\not{p}_4+m)\not{p}_3(\not{p}_2+m)\not{p}_3\right] \\
&= \frac{g^2}{2}\left(\mathrm{tr}[\not{p}_4\not{p}_3\not{p}_2\not{p}_3] + m^2\,\mathrm{tr}[\not{p}_3\not{p}_3]\right) = 4g^2(p_4\cdot p_3)(p_2\cdot p_3)\,.
\end{aligned} \tag{24}$$

In the second equality, we have used the usual techniques to take the conjugate and the amplitude. In the third, the identities for the sum over polarizations. Then we have developped the traces and used the trace identities, as well as $p_3 \cdot p_3 = 0$.

Let us now look at the kinematics in the center of mass frame:

$$\begin{aligned}
p_1^\mu &= (p,0,0,p), \qquad & p_2^\mu &= (E_p,0,0,-p), \\
p_3^\mu &= (p',\vec{p}\,'), \qquad & p_4^\mu &= (E_{p'},-\vec{p}\,'),
\end{aligned} \tag{25}$$

where $p' = |\vec{p}\,'|$ and $E_p = \sqrt{p^2+m^2}$. By conservation of energy, we find $p = p'$. We can compute

$$\begin{aligned}
p_3\cdot p_4 &= \frac{1}{2}\left((p_3+p_4)^2 - p_3^2 - p_4^2\right) = \frac{1}{2}\left((p+E_p)^2 - m^2\right) = p(p+E_p), \\
p_2\cdot p_3 &= pE_p + \vec{p}\cdot\vec{p}\,' = pE_p + p^2\cos\theta\,,
\end{aligned} \tag{26}$$

where θ is the angle between the final momentum $\vec{p}\,'$ and the z direction.

Then, we use the formula for cross-section

$$d\sigma = \frac{1}{2E_1 2E_2|v_1-v_2|}\left(\prod_{f=3,4}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\frac{1}{2}\sum_{r,s}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}\big(p_1+p_2-p_3-p_4\big)\,. \tag{27}$$

We can use the usual reduction of the 2-body phase space

$$\int\left(\prod_{f=3,4}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}\big(p_1+p_2-p_3-p_4\big) = \int\frac{d\Omega}{4\pi}\frac{1}{8\pi}\left(\frac{2|\vec{p}\,'|}{E_3+E_4}\right) = \int\frac{d\Omega}{4\pi}\frac{1}{8\pi}\left(\frac{2p}{p+E_p}\right)\,. \tag{28}$$

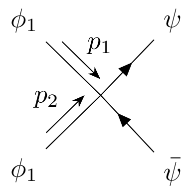
We also rewrite the form factor as

$$E_1E_2|v_1-v_2| = |E_2\vec{p}_1 - E_1\vec{p}_2| = p(p+E_p)\,. \tag{29}$$

We obtain the following expression, and the integral over the final particles angles is straightforward:

$$\begin{aligned}\sigma(\phi_1\psi \to \phi_2\psi) &= \frac{1}{4p(p+E_p)}\frac{1}{8\pi}\frac{2p}{p+E_p}4g^2p(p+E_p)\int\frac{d\Omega}{4\pi}(pE_p+p^2\cos\theta) \\ &= \frac{g^2p^2E_p}{4\pi(p+E_p)}\,.\end{aligned} \tag{30}$$

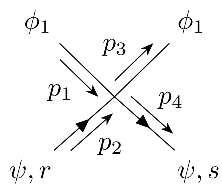
- If we now consider there is only one scalar field $\phi_1 = \phi_2$, in the feynman rules, one has to consider two possible ways to connect that field:



$$= \frac{-ig}{2}\big[(-ip_1)_\mu + (-ip_2)_\mu\big]\gamma^\mu = -\frac{g}{2}(\not{p}_1 + \not{p}_2)\,. \tag{31}$$

With this modification, one can see that all previous amplitudes vanish, using the equation of motion of the ψ field. For example:

$$i\mathcal{M}(\phi_1(p_1)\psi(p_2,r) \to \phi_1(p_3)\psi(p_4,s)) =$$



$$\begin{aligned} &= \bar{u}^s(p_4)g(-\not{p}_1 + \not{p}_3)u^r(p_2)\,, \\ &= g\bar{u}^s(p_4)(\not{p}_2 - \not{p}_4)u^r(p_2) = g\bar{u}^s(p_4)\big((\not{p}_2 - m) - (\not{p}_4 - m)\big)u^r(p_2) \\ &= 0\,, \end{aligned} \tag{32}$$

where in the last equality we have used $(\not{p} - m)u^r(p) = 0 = \bar{u}^s(p)(\not{p} - m)$. If we had not noticed the vanishing amplitude, we would anyway get 0 when doing the sum over polarizations of squared amplitudes.

Why is that 0 ? By using the product rule and integration by parts, we can get

$$\mathcal{L}_{int} = -\frac{g}{2}\partial_\mu(\phi_1\phi_1)\bar{\psi}\gamma^\mu\psi = \frac{g}{2}\phi_1\phi_1\partial_\mu(\bar{\psi}\gamma^\mu\psi) + tot.\ der. \tag{33}$$

We get an interaction term proportionnal to the divergence of the Noether current $\bar{\psi}\gamma^\mu\psi$ associated to the $U(1)$ symmetry of the spinor field. Since this current is conserved at leading order, this divergence vanishes up to terms $\mathcal{O}(g)$. Thus, the interaction is in g^2 at leading order and amplitudes would need to be computed at higher order in perturbation theory.

Exercise 3: Scalar QED Compton scattering

The Lagrangian from this theory is fixed by Lorentz invariance, hermiticity and gauge invariance and can be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - m^2|\phi|^2 - \frac{\lambda}{4}(|\phi|)^2 \tag{34}$$

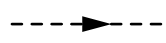
where the covariant derivative is

$$D_\mu = \partial_\mu + ieA_\mu. \tag{35}$$

This Lagrangian has three free parameters with the following energy dimension:

$$\begin{aligned}[m] &= 1 \\ [e] = [\lambda] &= 0\end{aligned} \tag{36}$$

We will now compute the Feynman rules for this theory. First of all we have the free field propagators for the scalar field



$$= \frac{i}{p^2 - m^2 + i\epsilon} \tag{37}$$

and the photon field (in the Feynman gauge)

μ ν

$$= \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}. \tag{38}$$

To find the Feynman rules for the interaction vertices we can compute some simple amplitudes. We will do it explicitly for one of the vertices. Expanding the kinetic term of the scalar field, we find an interaction between two scalars and the photon given by

$$\mathcal{L}_{int} = -ieA^\mu(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger). \tag{39}$$

To find the Feynman rule for this interaction, we can compute the transition amplitude between a state with a scalar and an anti-scalar and a photon. This amplitude actually doesn't contribute to the S matrix because in this process for on-shell particles it is impossible to conserve both energy and momentum. Anyway, this only means that the $\delta^{(4)}$ in front ot the amplitude will give zero when integrated on any initial and final wave-packets, but we can still compute the amplitude for the process.

The initial and final free states are given by

$$|i\rangle = |\phi\phi^\dagger\rangle = a^\dagger_{p_1} b^\dagger_{p_2} |0\rangle\,, \tag{40}$$

$$|f\rangle = |\gamma\rangle = c_k^{\dagger,1} |0\rangle\,. \tag{41}$$

where the creation operators are defined as

$$\phi = \int d\Omega_p (a_p e^{-ipx} + b_p^\dagger e^{ipx}) \tag{42}$$

$$A_\mu = \sum_{i=1}^{2} \int d\Omega_p (\epsilon^i_\mu(p) c^i_p e^{-ipx} + \epsilon^{*,i}_\mu(p) c^{\dagger,i}_p e^{ipx}). \tag{43}$$

The S matrix can be expanded up to the first order in perturbation theory as schematically [1]

$$S = Te^{i\int d^4x \mathcal{L}_{int}} \sim 1 + i\int d^4x\, \mathcal{L}_{int} = 1 + (2\pi)^4\delta^{(4)}(\textstyle\sum p) i\mathcal{M}. \tag{44}$$

So the amplitude is given by

$$(2\pi)^4\delta^{(4)}(\textstyle\sum p) i\mathcal{M} = \langle 0| c^1_k \left[i\int d^4x \left(-ieA^\mu(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger)\right)\right] a^\dagger_{p_1} b^\dagger_{p_2} |0\rangle\,. \tag{45}$$

To compute the amplitude we substitute the fields as in equation (42) to find

$$e\sum_{i=1}^{2}\int d^4x\, d\Omega_{q1} d\Omega_{q2} \Omega_{q3} \epsilon^{*,i}_\mu(q_3) i(q_2^\mu - q_1^\mu) \langle 0| (c^1_k c^{\dagger,i}_{q_3})(b_{q_2} b^\dagger_{p_2})(a_{q_1} a^\dagger_{p_1}) |0\rangle\, e^{ix(q_3 - q_1 - q_2)}. \tag{46}$$

This expression can be further simplified by using the commutation relations for the creation/annihilation operators

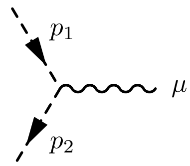
$$a_k a_q^\dagger |0\rangle = [a_k, a_q^\dagger] |0\rangle = (2\pi)^3 \sqrt{2E_k}\delta^{(3)}(\mathbf{k} - \mathbf{q}) |0\rangle\,. \tag{47}$$

[1] *Important note*: this derivation gives the correct result but is conceptually wrong. The reason is that we have interaction terms that contain derivatives and these are not easy to work with in the canonical quantization formalism. A correct derivation of these rules should be done using the path integral formalism.

This allows us to perform the integrals on the three momenta q_1, q_2 and q_3. Finally the integration on x gives us the energy/momentum conservation delta function. In the end we have

$$(2\pi)^4\delta^{(4)}(k-p_1-p_2)i\mathcal{M} = e(2\pi)^4\delta^{(4)}(k-p_1-p_2)\epsilon^{*,1}_{\mu}(k)\left[i(p_2^{\mu}-p_1^{\mu})\right]. \tag{48}$$

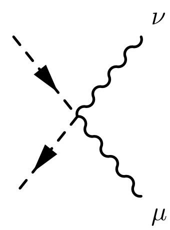
Factoring out the delta factor and the external polarization, we can read the Feynman rule for the vertex



$$= ie(p_2^{\mu}-p_1^{\mu}), \tag{49}$$

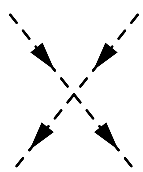
where in the Feynman rule the momentum are directed toward the vertex. When we use this rule we must be careful with the direction of the momentum with respect to the vertex and change sign if the momentum is outgoing.

The two other rules are found in a similar way: for the interaction between two scalars and two photons



$$= 2ie^2\eta^{\mu\nu}, \tag{50}$$

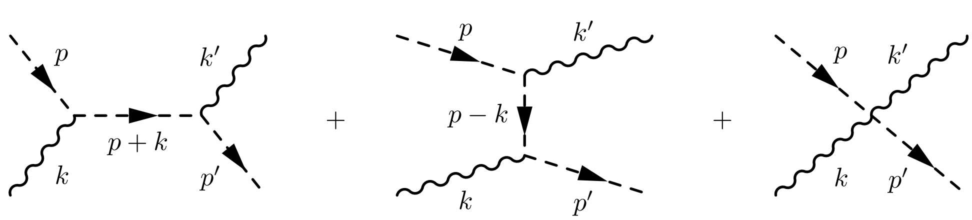
where the factor 2 comes from the two possible way of contracting the photon field with the external particles, and for the interaction between four scalars



$$= -i\lambda, \tag{51}$$

where the factor 1/4 cancels a factor 4 coming from the 2×2 possible way of contracting the field with the external legs.

We can now start the computation for the Compton scattering. We have three diagrams that contribute to this process



(52)

By using the Feynman rules written above we find the following amplitude

$$i\mathcal{M} = ie^2\epsilon^{i}_{\mu}(k)\epsilon'^{*,j}_{\nu}(k')\left[-\frac{(2p+k)^{\mu}(2p'+k')^{\nu}}{2p\cdot k}+\frac{(2p'-k)^{\mu}(2p-k')^{\nu}}{2p\cdot k'}+2\eta^{\mu\nu}\right]. \tag{53}$$

To verify that the Ward identity is satisfied, we need to show that the expression in the square brackets gives zero when contracted with the momenta of the polarization vectors. For example contracting with k_μ gives

$$\begin{aligned} k_{\mu}\mathcal{M}^{\mu\nu} \propto& k_{\mu}\left[-\frac{(2p+k)^{\mu}(2p'+k')^{\nu}}{2p\cdot k}+\frac{(2p'-k)^{\mu}(2p-k')^{\nu}}{2p\cdot k'}+2\eta^{\mu\nu}\right] \\ =& 2[p+k-p'-k'] = 0, \end{aligned} \tag{54}$$

where we have used $p \cdot k' = p' \cdot k$. Contracting with k'_ν is a similar calculation and gives zero as well.

In the laboratory frame we have the following momenta:

$$
\begin{aligned}
p &= (m, 0, 0, 0), && (55)\\
k &= E(1, 0, 0, 1), && (56)\\
k' &= E'(1, \sin\theta, 0, \cos\theta), && (57)\\
p' &= (m + E - E', -E'\sin\theta, 0, E - E'\cos\theta), && (58)
\end{aligned}
$$

where E' is given by the usual Compton formula

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m}(1 - \cos\theta). \tag{59}$$

To compute the differential cross section we need to compute the unpolarized squared modulus of the amplitude. There are two ways of doing this: the first one is to proceed as in the exercise of Compton scattering of a fermion, that is we use the trick of replacing the spin sum of photon polarization with $-\eta_{\mu\nu}$. This leads to some straightforward but lengthy algebra. A faster way to proceed is to compute explicitly $\mathcal{M}$ for the different polarization possibilities.

For the initial photon we chose the following base for the polarization vectors (transverse to the direction of motion to the photon)

$$
\begin{aligned}
\epsilon^1_\mu(k) &= (0, 1, 0, 0), && (60)\\
\epsilon^2_\mu(k) &= (0, 0, 1, 0). && (61)
\end{aligned}
$$

For the final photon we rotate these two polarization vectors to be transverse to the direction of motion of the final photon

$$
\begin{aligned}
\epsilon'^1_\mu(k') &= (0, -\cos\theta, 0, \sin\theta), && (62)\\
\epsilon'^2_\mu(k') &= (0, 0, 1, 0). && (63)
\end{aligned}
$$

The advantage of this choice is that most of the scalar products are zero. In fact we have:

$$p \cdot \epsilon = k \cdot \epsilon = p \cdot \epsilon' = k' \cdot \epsilon' = 0, \tag{64}$$

meaning that the only non-zero contribution to the amplitude comes from the third diagram

$$i\mathcal{M} = 2ie^2(\epsilon^i \cdot \epsilon'^{,j}) \tag{65}$$

and we immediately find

$$\frac{1}{2}\sum_{i,j=1}^{2} |\mathcal{M}|^2 = 2e^4(1 + \cos^2\theta). \tag{66}$$

To compute the cross section we need the flux factor

$$F = p \cdot k = mE \tag{67}$$

and the phase-space measure

$$d\Phi_2 = \frac{1}{16\pi^2}\frac{(E')^2}{mE}d\Omega. \tag{68}$$

The differential cross section is then given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\frac{1 + \cos^2\theta}{(1 + E/m(1 - \cos\theta))^2} \tag{69}$$

that in the low energy limit $E \ll m$ reduces to the usual Thomson formula

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}(1 + \cos^2\theta) \tag{70}$$

that can be easily integrated to find the Thomson cross-section

$$\sigma = \frac{8\pi\alpha^2}{3m^2}. \tag{71}$$

Exercise 4

For convenience we start by writing the interaction part of the Lagrangian as

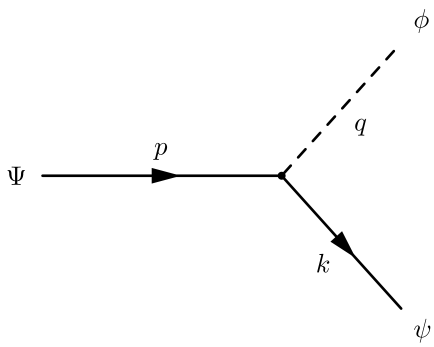
$$\mathcal{L}_{\text{int}} = \lambda\bar{\psi}P_R\Psi\phi + \lambda\bar{\Psi}P_L\psi\phi \tag{72}$$

where Ψ is a Dirac field of mass M while ψ is a massless Dirac field and ϕ is a massless real scalar.

Each fermion has energy dimension 3/2 and the scalar has dimension 1. Thus, $\bar{\psi}_L\Psi_R\phi$ has dimension 4, meaning that λ must be dimensionless.

Now consider the process $\Psi \to \psi, \phi$ where Ψ has momentum p and polarization r, ψ has momentum k and polarization s while ϕ has momentum q. It is straight-forward to derive the Feynman rule

$i\mathcal{M} =$



$$= i\lambda\bar{u}_s(k)P_R u_r(p) \tag{73}$$

Now to compute $\mathcal{M}^*$ we need

$$(\bar{u}_s(k)P_R u_r(p))^* = u_r(p)^\dagger P_R^\dagger \gamma_0^\dagger u_s(k) = \bar{u}_r(p)P_L u_s(k) \tag{74}$$

where we used $\gamma_0^\dagger = \gamma_0$ and $P_R\gamma_0 = \gamma_0 P_L$. Thus, the matrix element squared averaged over the initial polarizations and summed over the final polarizations reads

$$\frac{1}{2}\sum_{r,s}|\mathcal{M}|^2 = \frac{1}{2}\sum_{r,s}\lambda^2\bar{u}_r(p)P_L u_s(k)\bar{u}_s(k)P_R u_r(p) \tag{75}$$

Now using $\sum_s u_s(k)\bar{u}_s(k) = \not{k}$ and $\sum_r u_r(p)\bar{u}_r(p) = \not{p} + M$ (recall that Ψ has mass M while ψ is massless) we obtain

$$\frac{1}{2}\sum_{r,s}|\mathcal{M}|^2 = \frac{\lambda^2}{2}\text{Tr}((\not{p}+M)P_L\not{k}P_R) = \frac{\lambda^2}{2}\text{Tr}((\not{p}+M)\not{k}P_R) \tag{76}$$

Now note that only $\not{p}\not{k}\frac{1}{2}\mathbb{1}$ contributes to the trace. Thus,

$$\frac{1}{2}\sum_{r,s}|\mathcal{M}|^2 = \lambda^2 p\cdot k = \frac{\lambda^2 M^2}{2} \tag{77}$$

where in the last equality we used $0 = q^2 = (p-k)^2 = M^2 - 2p\cdot k$ by momentum conservation which implies $p\cdot k = M^2/2$. To compute the decay rate we use the general formula for a decay $A \to CD$

$$d\Gamma_{A\to CD} = \frac{1}{2M}|\mathcal{M}|^2\frac{d\varphi d\cos\theta}{16\pi^2}\frac{p_c(M)}{M} \tag{78}$$

where $p_C(M)$ is the norm of the 3-momentum of particle C. Here by momentum conservation and the fact that the final particles are massless, $|\vec{p}_C| = |\vec{k}| = M/2$. Moreover, since we are averaging over initial polarizations and summing over all final polarizations we should replace $|\mathcal{M}|^2 \to \frac{1}{2}\sum_{r,s}|\mathcal{M}|^2$. Finally, our matrix elements is rotationally invariant so the integral over the solid angle results in a factor 4π. Altogether we obtain

$$\Gamma = \frac{\lambda^2 M}{32\pi} \tag{79}$$

Note that the dependence in λ and M could have been guessed by a quick analysis. First, we know that the interaction vertex goes as λ. Thus, $\Gamma \propto |\mathcal{M}|^2 \propto \lambda^2$. Moreover, we know that Γ should have energy dimension 1. The only dimensionfull scale at hand is the mass M of the Ψ particle. Thus the only possibility at leading order in λ is $\Gamma \sim \lambda^2 M$ as we derived.